Calculating residues. Hurwitz Theorem.

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How to calculate residues?

Case 1. Simple pole at z_0.

$$f(z) = \frac{g(z)}{z-z_0}, \quad g \in A(B(z_0,r)), \quad g(z_0) \neq 0.$$

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\oint_{C_r} f(z)\,dz = \frac{1}{2\pi i}\oint_{C_r} \frac{g(z)}{z-z_0}\,dz = g(z_0) = \boxed{\lim_{z\to z_0}(z-z_0)f(z)}$$

And if $f \in A(B(z_0,r)\setminus\{z_0\})$, and $\exists \lim_{z\to z_0}(z-z_0)f(z) =: R$,

then $g(z) = \begin{cases} f(z)(z-z_0), & z \neq z_0 \\ R, & z = z_0 \end{cases} \in A(B(z_0,r))$. So $R = \operatorname{Res}_{z=z_0} f(z)$, as above.

Example. $\operatorname{Res}_{z=0} \frac{1}{\sin z} = \lim_{z\to 0}\frac{z}{\sin z} = \frac{1}{\lim_{z\to 0}\frac{\sin z - \sin 0}{z-0}} = \frac{1}{\cos 0} = 1.$

If $f(z) = \frac{g(z)}{h(z)}$, $\quad g,h \in A(B(z_0,r))$, $\quad h(z_0)=0$, $h'(z_0)\neq 0$ (simple zero).

Then $\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0}\frac{g(z)}{h(z)}\cdot(z-z_0) = \frac{\lim_{z\to z_0} g(z)}{\lim_{z\to z_0}\frac{h(z)-h(z_0)}{z-z_0}} = \boxed{\frac{g(z_0)}{h'(z_0)}}$

Example $\operatorname{Res}_{z=0}\cot z = \frac{\cos 0}{\sin' 0} = 1.$

Case 2. f has a pole of order h at z_0.

$$f(z) = \frac{g(z)}{(z-z_0)^h}, \text{ where } g(z_0)\neq 0, \quad g \in A(B(z_0,r)).$$

By Cauchy:

$$g^{(h-1)}(z_0) = (h-1)!\,\frac{1}{2\pi i}\oint_{C_\delta}\frac{g(z)}{(z-z_0)^h}\,dz = (h-1)!\,\frac{1}{2\pi i}\oint_{C_\delta} f(z)\,dz = \operatorname{Res}_{z=z_0} f(z)\,(h-1)!$$

So

So $$\boxed{\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(k-1)!}\left(\frac{d^{k-1}}{dz^{k-1}}\left((z-a)^k f(z)\right)\right).}$$

Case 3. Essential singularity: no good formula.

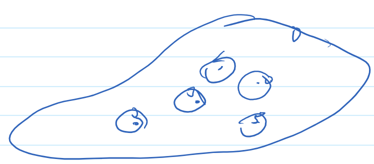
General argument principle.

Theorem. Let $f \in \mathcal{M}(\Omega)$. $\gamma \subset \Omega$ - cycle, $\gamma \sim 0$ in Ω.

$\forall z \in \gamma$, $f(z) \neq 0, \infty$ (no poles or zeroes on γ)

Then $$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{f(z)=0} n(\gamma, z)\,\mathrm{ord}(f,z) + \sum_{f(z)=\infty} n(\gamma,z)\,\mathrm{ord}(f,z).$$

Reminder: "Argument principle" because
$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = n(f\circ\gamma, 0).$$



Remark. As usual, there are only finitely many zeroes and poles of f for which $n(\gamma, z) \neq 0$, so both sums on RHS are finite.

Proof. $\frac{f'(z)}{f(z)} \in A(\Omega \setminus \{z : f(z)=0 \text{ or } f(z)=\infty\})$

So, by residue theorem
$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{f(w)=0\\ f(w)=\infty}} n(\gamma, z)\,\mathrm{Res}_{z=w}\frac{f'(z)}{f(z)}.$$

Let $\mathrm{ord}(f, w) = h \neq 0$ (w is a zero or pole).

Then $f(z) = (z-w)^h f_1(z)$, where $f_1(w) \neq 0$, $f_1 \in A(B(w,\delta))$ for some $\delta > 0$.

$$\frac{f'(z)}{f(z)} = \frac{h(z-w)^{h-1}f_1(z) + (z-w)^h f_1'(z)}{(z-w)^h f_1(z)} = \frac{h}{z-w} + \frac{f_1'(z)}{f_1(z)}.$$ (we also knew it by properties

$$\frac{f'(z)}{f(z)} = \frac{\cdots}{(z-w)^h f_1(z)} = \frac{h}{z-w} + \frac{f_1'(z)}{f_1(z)}.$$ (we also knew it by properties of logarithmic derivative).

So $$\mathrm{Res}_{z=w} \frac{f'(z)}{f(z)} = \lim_{z \to w} (z-w)\left(\frac{h}{z-w} + \frac{f_1'(z)}{f_1(z)}\right) = h + 0 \cdot \frac{f_1'(w)}{f_1(w)} = h = \mathrm{ord}(f, w).$$ ∎

<u>Corollary</u> Let γ be an oriented boundary of Ω, $f \in \mathcal{M}(\Omega \cup \gamma)$, $f(z) \neq 0$ and $f(z) \neq \infty$ if $z \in \gamma$.

Then
$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz = N - P,$$ where N is the number of zeroes of f in Ω (counted with multiplicity), P – the number of poles in Ω (also counted with multiplicity).

<u>Theorem</u> (Rouche) Let $f, g \in A(\Omega)$, γ – simple closed

curve in Ω, $\gamma \sim 0$ in Ω. Assume
$\forall z \in \gamma \quad |f(z) - g(z)| < |f(z)|$.
Then f and g have the same number of zeroes (N_f and N_g) inside of γ, counted with multiplicities.

<u>Proof</u> The same as for the local version.

Adolf Hurwitz

<u>Theorem</u> (Hurwitz).
Assume $f_n \in A(\Omega)$, $\forall z \in \Omega \; f_n(z) \neq 0$. Let $f_n \to f$ locally uniformly.
Then $\forall z \in \Omega \; f(z) \neq 0$ <u>or</u> $f(z) \equiv 0$.

<u>Proof</u>. By Weierstrass Theorem, $f \in A(\Omega)$.
~~Assume~~: $f \not\equiv 0$, $f(z_0) = 0$. Then $\exists r > 0$:
$0 < |z - z_0| \leq r \Rightarrow z \in \Omega, \; f(z) \neq 0$ (zeroes are isolated).
Let $C_r = \{ |z - z_0| = r \}$. Then $m = \min_{z \in C_r} |f(z)| > 0$.
$f_n \rightrightarrows f$ uniformly on C_r. Take k: $|f_k(z) - f(z)| < m$
$\forall z \in C_r$.

Then, by Rouche, f_k and f have the same number of zeroes inside C_r ($|f_k(z) - f(z)| < m \leq |f(z)|$). But $f_k(z) \neq 0 \; \forall z$, $f(z_0) = 0$ -
contradiction! ∎

<u>Corollary</u>. $f_n \in A(\Omega)$, injective (= conformal).
$f_n \to f$ locally uniformly on Ω. Then either f is conformal or $f \equiv$ const.

Proof

Assume $f \not\equiv$ const.

Fix $z_0 \in \Omega$. Consider $g_n(z) := f_n(z) - f_n(z_0) \neq 0$ in $\Omega \setminus \{z_0\}$.

$g_n(z) \to f(z) - f(z_0)$ locally uniformly, $g_n(z) \neq 0$ in $\Omega \setminus \{z_0\}$.

So, by Hurwitz, $f(z) \neq f(z_0)$ $\forall$ $z \in \Omega \setminus \{z_0\}$.

So for $z \neq z_0$, $f(z) \neq f(z_0)$ - injective $\blacksquare$